The concept of dimension

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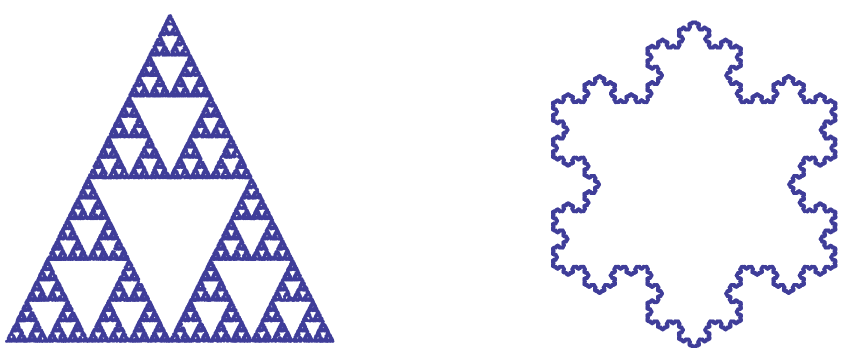
There are several contexts in mathematics where we need the concept of dimension. One of them is the study of fractals. Indeed, in fractal geometry we encounter very complex objects. We must find a way to quantify their complexity and dimension provides a measure of their complexity. We will discuss some ways to describe fractal objects by working on two examples: the Sierpinski carpet and the von Koch flake (see Figure 1).

1 What is the area of the Sierpinski carpet?

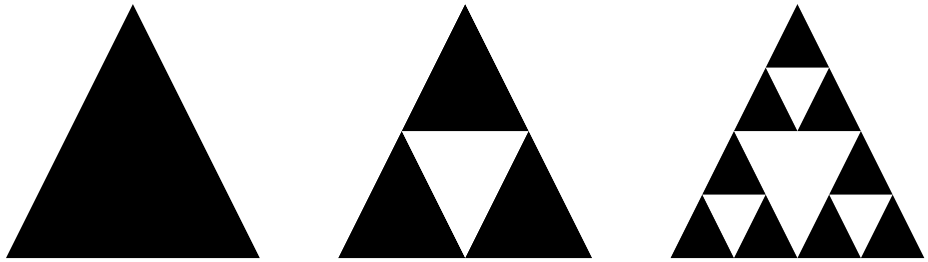
We first need to understand the construction of the Sierpinski carpet (see Figure 2). It is done through an iterative process. We start with a triangle and, at each step, we remove the middle triangle. We are left with three triangles. Within each remaining triangle we remove the middle triangle, etc. We now have all the ingredients to compute the area of the Sierpinski carpet. Let us suppose that the area of the initial triangle (see Figure 2(a)) is A.

- At the first iteration we remove an area of $\frac{A}{4}$ and we are left with an area of $A_1 = \frac{3A}{4}$.
- At the second iteration, we remove one fourth of the area of the three remaining triangles, so one fourth of A_1. Hence, the remaining area is $A_2 = \frac{3}{4}A_1 = \left(\frac{3}{4}\right)^2 A$.
- At the third iteration, we remove one fourth of the area of the nine remaining triangles, so one fourth of A_2. Hence, the remaining area is $A_3 = \frac{3}{4}A_2 = \left(\frac{3}{4}\right)^3 A$.



(a) Sierpinski carpet (b) Von Koch Flake

Figure 1: Sierpinski carpet and Von Koch flake.



(a) initial triangle (b) First iterate (c) Second iterate

Figure 2: The iteration process to construct the Sierpinski carpet.

- ...
- At the n-th iteration we remove one fourth of the area of the 3^{n-1} remaining triangles, so one fourth of A_{n-1}. Hence, the remaining area is $A_n = \frac{3}{4}A_{n-1} = \left(\frac{3}{4}\right)^n A$.
- ...

Since
$$\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0,$$
we can conclude that the area of the Sierpinski carpet is zero!

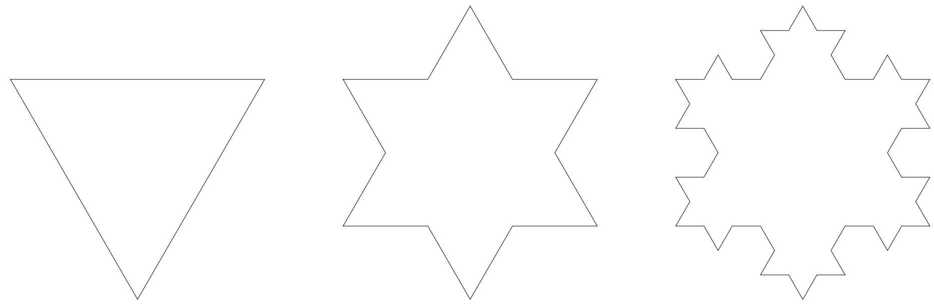
2 What is the length of the von Koch flake?

Again, the von Koch flake is obtained by iteration. At each step of the iteration, we replace each segment by a group of 4 segments with length equal to 1/3 of the length of the original segment (see Figure 3). If L is the length of the original triangle in Figure 3(a), then $\frac{4}{3}L$ is the length of the star of Figure 3(b), $\left(\frac{4}{3}\right)^2$ is the length of the object in Figure 3(c), etc. In particular, this means that, at each step, the length is multiplied by $\frac{4}{3}$. Since there are an infinite number of steps in the construction, then the length of the von Kock flake is infinite!

3 Dimension of a fractal object

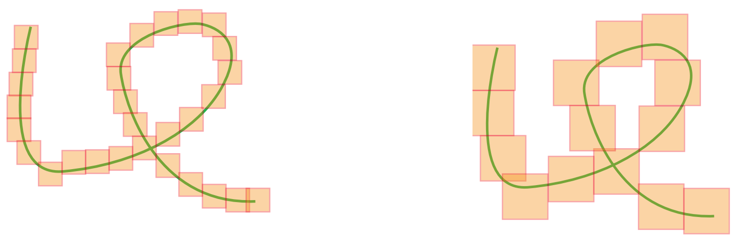
The Sierpinski carpet is a very complicated object. Nevertheless, its area is zero and hence, gives us little information on the object. The fact that the length of the von Koch flake is infinite tells us that the object is complicated, but no more precision. To be able to give more information on fractal objects the mathematicians introduce the concept of *dimension*.

How does a mathematician give a definition of dimension? We start with our intuitive idea of dimension. Intuitively, smooth curves are of dimension 1, smooth surfaces of dimension 2, and filled volumes of dimension 3. So we should give a mathematical definition of dimension which yields 1 for smooth curves, 2 for smooth surfaces, and 3 for filled volumes. In the context of this vignette we will limit ourselves to dimensions 1 and 2. We want to cover a



(a) Initial triangle (b) First iterate (c) Second iterate

Figure 3: Von Koch flake and the iteration process to construct it.



(a) (b)

Figure 4: Calculating the dimension of a curve using squares of different sizes.

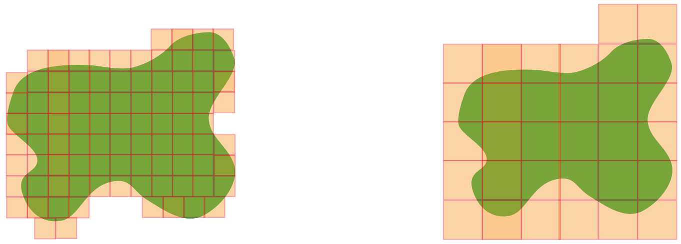
geometric object in the plane with small squares. (If we would want to define dimension 3 we would use small cubes, but we could have used small cubes for curves and surfaces without changing the dimension!)

Case of a smooth curve. (see Figure 4)

- if we take squares with side of half size, then we approximately double the number of squares needed to cover the object;
- if we take squares with side one third of the size, then we approximately triple the number of squares needed to cover the object;
- ...
- if we take squares with side n times smaller, then we approximately need n times more squares to cover the object;

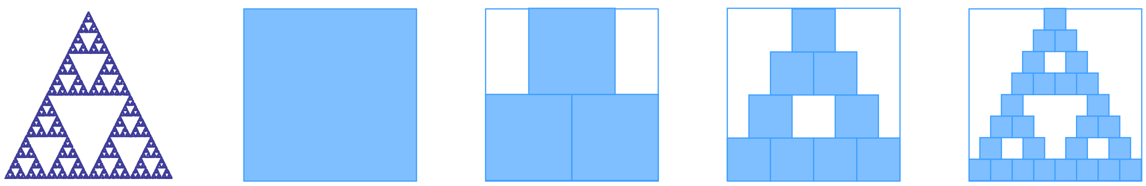
Case of a surface. (see Figure 5)

- if we take squares with side of half size, then we approximately need four times more squares to cover the object;



(a) (b)

Figure 5: Calculating the dimension of a surface using squares of different sizes.



(a) Sierpinski triangle (b) 1 (c) 3 (d) 9 (e) 27

Figure 6: Number of squares to cover to cover the Sierpinski carpet appearing in (a).

- if we take squares with side one third of the size, then we approximately need nine times more squares to cover the object;
- ...
- if we take squares with side n times smaller, then we approximately need n^2 times more squares to cover the object;

We can now give the (intuitive) definition of dimension:

Definition 3.1 An object has dimension d if, when we take squares (or cubes) with edge n times smaller to cover it, then we need approximately n^d more squares to cover it.

Not all objects have a dimension. But, the self similar objects have a dimension which, most often, is not an integer. Let us now calculate the dimension of the Sierpinski carpet (see Figure 6).

- Let us take a square with side equal to the length of the base. It covers the Sierpinski carpet of Figure 6(a).
- if we take squares with side of half size, then we need three squares to cover it. Note that $3 = 2^{\frac{\ln 3}{\ln 2}}$ (Figure 6(c)).
- if we take squares with side one fourth of the size, then we need nine squares to cover it. Note that $9 = 4^{\frac{\ln 3}{\ln 2}}$ (Figure 6(d)).

- if we take squares with side one eight of the size, then we need 27 squares to cover it. Note that $27 = 8^{\frac{\ln 3}{\ln 2}}$ (Figure 6(e)).

So, it is easy to conclude that the dimension of the Sierpinski triangle of Figure 6(a) is $d = \frac{\ln 3}{\ln 2} \sim 1.585$.

We assert without proof that it is also the dimension of the von Koch flake of Figure 1(b) is $\frac{\ln 4}{\ln 3} \sim 1.26$. The calculation is more difficult to do than for the Sierpinski carpet.

The dimension gives a "measure" of the complexity or density of a fractal. Indeed, we feel that the Sierpinski carpet is denser than the Von Koch flake, which looks more like a thickened curve. This is reflected by the fact that $\frac{\ln 3}{\ln 2} > \frac{\ln 4}{\ln 3}$.

3.1 Applications

The capillary network is not the same in the neighborhood of a tumor as elsewhere in the body. Research is carried on this, and in particular on its fractal dimension, in order to improve diagnosis from medical imaging.

High level athletes are more likely to suffer from asthma than the general population. The paper [1] studied the "optimal lung. There are 17 level of bronchial tubes before arriving to the terminal bronchioles followed by the acini involved in air exchange. If the bronchial tubes are too thin, then the pressure increases when the air penetrates in the next level of bronchial tubes. But it they are too wide, so that the volume remains the same at each level, then the volume becomes too large. (It would become infinite if we had an infinite number of levels). So, the "optimal"lung would have the minimum volume without increasing the pressure. But the curves giving the pressure and volume for a given homothety ratio from one level of bronchial tree to the next are nonlinear and the rate of variation of pressure close to the optimal lung is quite high. The human lungs have a higher volume than the theoretical optimal lung. This buffer provides a protection in case of bronchoconstriction, a pathology decreasing the diameter of the bronchial tubes, which could be caused by asthma. Athletes have lungs generally closer to the theoretical optimal lung, and are hence more vulnerable.

The outer surface of the small intestin has an approximate area of 0.5 m^2, while the inner surface has an approximate area of 300 m^2. We have seen with the von Koch flake that a fractal curve can have an infinite length, even if it lies in a finite surface. Similarly, we could easily imagine that a fractal surface lying in a finite volume can have an infinite area. This is a trick used by nature: the area of the inner surface of the small intestin must be very large in order to maximize intestinal absorption. The fractal nature of this surface achieves this goal. The same is true for the surface of the alveoli at the end of the bronchioles in the lungs. Since the bronchial tree has a fractal nature, the surface of the alveoli is extremely large, thus maximizing the gas exchange.

References

[1] B. Mauroy, M. Filoche, E.R. Weibel, B. Sapoval, An optimal bronchial tree may be dangerous, Nature, **427** (2004), 633–636.